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CIRCULAR COORDINATES.

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1. BY an extension of the idea of the "position ratio" of a point, referred to two fundamentel points A and B, any point in a plane may be determined by the complex ratio of its position ratio. Put

$$\frac{AP}{BP} = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta) = x + iy; \qquad (1)$$

then, if r and r' are the lengths of the lines AP and BP, the usual interpretation of the ratio of directed lines gives

$$\rho = \frac{r}{r'}, \qquad \text{and} \qquad \theta = BPA.$$

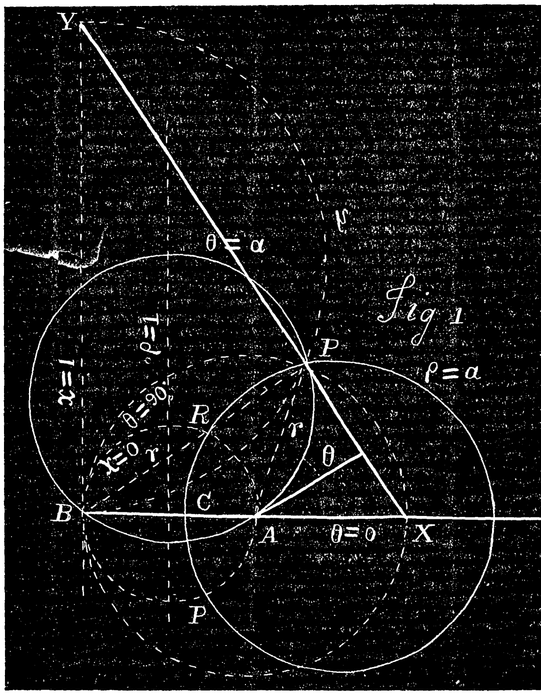
The real quantities ρ and θ may be taken as a system of coordinates analogous to the ordinary polar system; and x and y may be taken as a system of coordinates bearing to this last system the same relations [see eq. (1)],

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

that connect the ordinary rectangular and polar coordinates.

2. The locus of $\theta = a$ is obviously a circle passing through A and B, in one segment of which (say the upper segment in Fig. 1) the angle BPA, considered as reckoned from PB towards PA, is a, and in the other segm't ρ must be taken as negative and $\theta = a$, or else ρ is positive and $\theta = \pi + a$. The locus of $\rho = a$ is also a circle, being the locus for which the ratio $r \div r'$ is constant, that is, the circle described on CD as a diameter, where C and D cut AB harmonically in the ratio ρ. These circles cut at right angles; for, if O is the centre of the latter, $OA.OB = OC^2 = OP^2$, hence OP is tangent to the circle BAP. Thus, in the $\rho\ \theta$ system, a point P is determined as the intersection of two circles which cut at right angles, the second point of intersection P', being distinguished from P either by regarding ρ as negative or by adding 180° to θ.

3. If we draw a tangent to the θ-circle at A, θ is the inclination of this tangent to the direction BA produced. θ is indeterminate at each of the points B and A; $\theta=0$ represents the infinite circle through $B\ A$; that is, $\theta\ =\ 0$ not only on the straight line BA, but at every infinitely distant p't $\rho=0$ and ∞ respectively at A and B, the points where θ is indeterminate. $\rho=1$ is the equation of the straight line bisecting BA at right angles, and, since this is to be regarded as an infinite circle, $\rho=1$ at every infinitely distant point We should however in this system speak of *the point* and not *the line at infinity*, and the coordinates of this point are $\rho=1$, $\theta=0$.



4. If the fundamental point B be removed to infinity, the θ-circle becomes a straight line making an angle θ with the axis BA, and the ρ-circle ultimately has A for its centre; ρ becomes infinitesimal, but multiplying it by the constant infinite factor r', we have the ordinary polar coordinates.

5. Proceeding now to the $x\ y$ coordinates, the locus of $x\ =$ a constant is also a circle; for, if the circle whose diameter is BA cuts r' in R, $RP=r\cos\theta$; hence if

$$x=\rho\cos\theta=\frac{r\cos\theta}{r'}=\frac{RP}{BP} \tag{2}$$

be constant, the line BP bears a constant ratio to BR, and P describes a locus similar to that of R, in other words, a circle whose diameter is BX, where X is that point of BA whose position ratio is the given value of x, since at X $x=\rho=AX\div BX$. Since

$$1-x=\frac{BX-AX}{BX}=\frac{BA}{BX},$$

the diameter of this circle is

$$BX=\frac{BA}{1-x}=\frac{c}{1-x}. \tag{3}$$

The locus of $y =$ a constant is also a circle; for let

$$y = \rho \sin \theta = \frac{r \sin \theta}{r'} = \frac{AR}{BP} \tag{4}$$

be constant; then since AR is perpendicular to BP, a point R' on BR such that $BR' = AR$ will describe a circle equal to BRA and touching BA; and, BP having a constant ratio to BR', P will describe a similar curve. If this circle cut the perpendicular to BA through B in Y, eq'n (4) gives

$$y = \frac{AB}{BY},$$

hence the diameter of this circle is

$$BY = \frac{BA}{y} = \frac{c}{y}. \tag{5}$$

The x- and y-circles cut at right angles; thus the x y coordinates determine a point as the variable intersection of two circles which cut also in the fixed point B. When B is removed to infinity, the arcs PX and RA become straight lines perpendicular to BA, and PB, a straight line parallel to BA; the values of x and y vanish, but being multiplied by the infinite factor r', become RP and AR [equations (2) and (4)] which are now ordinary rectangular coordinates.

6. Negative values of x correspond to points within the circle $x = 0$, or $\theta = 90°$, whose diameter is AB. $x = 1$ is the equation of the perpendicular BY, but $x = 1$ also at every infinitely distant point; the coordinates of the "point at infinity" are $x = 1$, $y = 0$.

7. If we put $1 - x = x_1$, equations (3) and (5) become

$$x_1 = \frac{c}{BX}, \qquad y = \frac{c}{BY}, \tag{6}$$

and the coordinates (x_1, y) are the reciprocals of the diameters of the circles. Dropping the suffix, $x = 0$ and $y = 0$ now represent the rectangular axes intersecting at B, but their intersection is not the point $(0, 0)$, since the p't (x, y) is defined as the variable intersection of the circles, which is now the point at infinity.

8. It is evident that P is the foot of a perpendicular from B upon XY; x and y, the reciprocals of the intercepts upon the axes, are Dr. Booth's tangential coordinates of this line, and they are the ordinary rectangular coordinates of the pole of this line with respect to B, which is the inverse point to P, B being the centre of inversion. In fact, if x', y' are the rectangular coordinates of P we have $x'.BX = BP^2 = x'^2 + y'^2$, and, comparing with (6),

$$x = \frac{cx'}{x'^2 + y'^2}, \text{ and similarly } y = \frac{cy'}{x'^2 + y'^2}, \tag{8}$$

which are also the relations between the coordinates of inverse points. Thus the circular coordinates of P are the rectangular coordinates of the inverse point.

9. It follows that the equation of a curve in circular coordinates is the same as that of its inverse in rectangular coordinates. Thus the equation of the first degree in circular coordinates may be written in the form

$$mx + ny + c = 0;$$

the corresponding rectangular equation is

$$x'^2 + y'^2 + mx' + ny' = 0,$$

showing that the locus is a circle passing through B, and making BA an angle whose tangent is $-(n \div m)$. On the other hand, the equation

$$x^2 + y^2 + mx + ny = 0$$

in circular coordinates is the general equation of the sraight line not passing through B; while the general equation of the circle retains the same form as in rectangular coordinates.

10. Let us now suppose that the position of P is determined not simply by its position ratio, but by the ratio of its position ratio to that of a third fixed point Q referred to A and B, that is, by the anharmonic ratio

$$\frac{P\ A}{B\ Q} = \frac{AP}{BP} : \frac{AQ}{BQ}.$$

Denoting as before the position ratio of P by $\rho e^{i\theta}$, and denoting t at of Q by $\rho_0 e^{i\theta_0}$, we have for the value of this expression,

$$\frac{P\ A}{B\ Q} = \frac{\rho}{\rho_0} e^{i(\theta-\theta_0)} = \frac{x + iy}{x_0 + iy_0},$$

and if we put

$$\frac{P\ A}{B\ Q} = Re^{i\Theta} = X + iY,$$

it is evident that the locus of R = constant and of Θ = constant are circles of the ρ- and θ-systems respectively, but R is a fixed multiple of the ratio of the lengths of AP and BP, and Θ is the angle between the tangents at A of the circles APB and AQB.

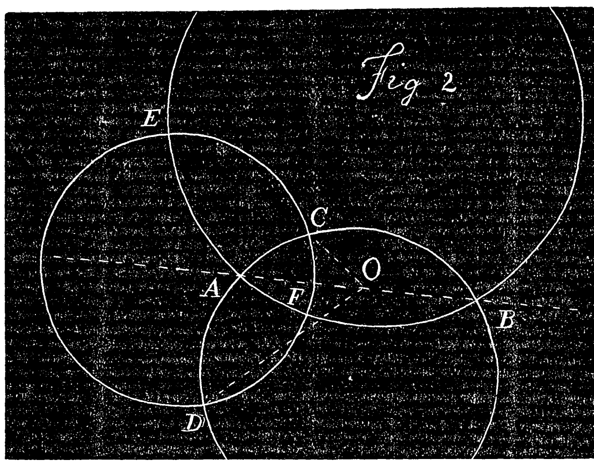
Since $$X+iY = \frac{x + iy}{x_0+iy_0} = \frac{xx_0+yy_0}{x_0^2 + y_0^2} = i\frac{yx_0-xy_0}{x_0^2 + y_0^2},$$

the locus of X= a constant is, by § 9, a circle passing through B and touching at that point a fixed line which makes with BA the angle whose tang't is $-(x_0 \div y_0)$. In like manner, the locus of Y = a constant is a circle which touches at B a line inclined to BA at an angle whose tangent is $y_0 \div x_0$; that is, a line perpendicular to that touched by the X-circles. Furthermore, a linear relation between X and Y is equivalent to a linear relation

between x and y and represents a circle passing through B. In fact, the system $X\ Y$ bears to the system $x\ y$ the same relations that connect two ordinary rectangular systems with the same origin, to which systems indeed they reduce when B is removed to infinity.

11. Four points on the same circle have a real anharmonic ratio. In particular the anharmonic ratio of the section of AB by PP' in Fig. 1 is —1. The points P, P' are therefore harmonic conjugates with respect to A, B, and reciprocally A, B are harmonic conjugates with respect to P, P'; that is, *when two circles cut orthogonally, the points in which a diameter of one cuts the circumference of the other are harmonic conjugates with respect to the points of intersection.*

12. The harmonic cycle of points AB, CD has also the following property analogous to a property of the harmonic range; viz., *if O is the middle point of AB, OC and OD make equal angles with OA and $OC.OD = OA^2$.*



13. *If three circles cut each other orthogonally*, as in Fig. 2, *their points of intersection form three pairs of points AB, CD and EF each of which is a pair of harmonic conjugates with respect to either of the other pairs;* for the centre of each circle is on the radial axis of the other two.

In the same figure the position ratios of C and E, with respect to AB, are equal in modulus and their arguments differ by 90°, hence their ratio is i or $-i$; the six anharmonic ratios of four points thus situated being

$$i, \quad -i, \quad 1+i, \quad \frac{1-i}{2}, \quad \frac{1+i}{2}, \quad 1-i.$$

14. The real anharmonic ratio of four points on the circumference of a circle is the same as the anharmonic ratio of the pencil formed by joining the points to any point of the circumference, a property by which the anharmonic ratio of four points on a circumference is generally defined. But the anharmonic ratio of the concyclic points is not the same as that of the pencil formed by joining the points to *any* point as in the case of the rectilinear range; but it is to be noticed that when the radius of the circle

becomes infinite, any finite point may be regarded as at an infinitesimal distance from the circumference relatively to the infinite radius.

15. There is a special case of the complex anharmonic ratio which is worthy of notice, in which, as in the case of the harmonic ratio, there are less than six *different* values of the anharmonic ratio. In the harmonic ratio, there are in fact three different values, viz., -1, 2, and $\frac{1}{2}$, the first of these numbers being its own reciprocal, the next being (in the nomenclature of my articles, ANALYST Vol. IX, p. 185, and Vol. X, p. 76) its own conjugate, and the third its own complement. The only other case in which there are but three different values is that of 1, 0, and ∞, which occurs when two of the four points are coincident. In the case in question however there are but two different values, namely $\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$, each of which is at once the reciprocal, complement and conjugate of the other. If the fundamental points A, B and C are given, two points P and P' may be found such that

$$\begin{matrix} P & A \\ B & C \end{matrix} \quad \text{and} \quad \begin{matrix} P' & A \\ B & C \end{matrix}$$

shall have these two values, and then each of the anharmonic ratios of the points $ABCP$ or $ABCP'$ will have one of the two values, so that the interchange of any two points of the four has the same effect. Since for these values, which are $e^{\pm\frac{1}{3}i\pi}$, $R=1$ (see § 10) $\rho = \rho_0$; hence, denoting the sides of the triangle by a, b, c and the distances from P to the vertices A, B, C by a', b' and c', we have $aa' = bb' = cc'$; and, since $\theta = \pm\frac{1}{3}\pi$, the difference between the angles BCA and BPA (reckoned in the same direction), QAP and QBP, etc., is in each case equal to 60°.

If ABC is an isosceles triangle, P and P' are on the bisector of the angle opposite the base. If one of the angles of the triangle is 60° or 120°, one of the points is on the side opposite this angle; but if the triangle is equilateral this point is at infinity, the other being the centre of the triangle.

ON THE DIVISIBILITY OR NON-DIVISIBILITY OF NUMBEBS BY SEVEN.

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THE test for divisibility by seven has been considered in relation to the number of digits composing the dividend: As "for two or three figures; for three or four figures; for five figures; for five, six, seven or more figures."

A writer, after so treating the subject, says, "the various tokens of seven are not so difficult to remember as at first may appear".